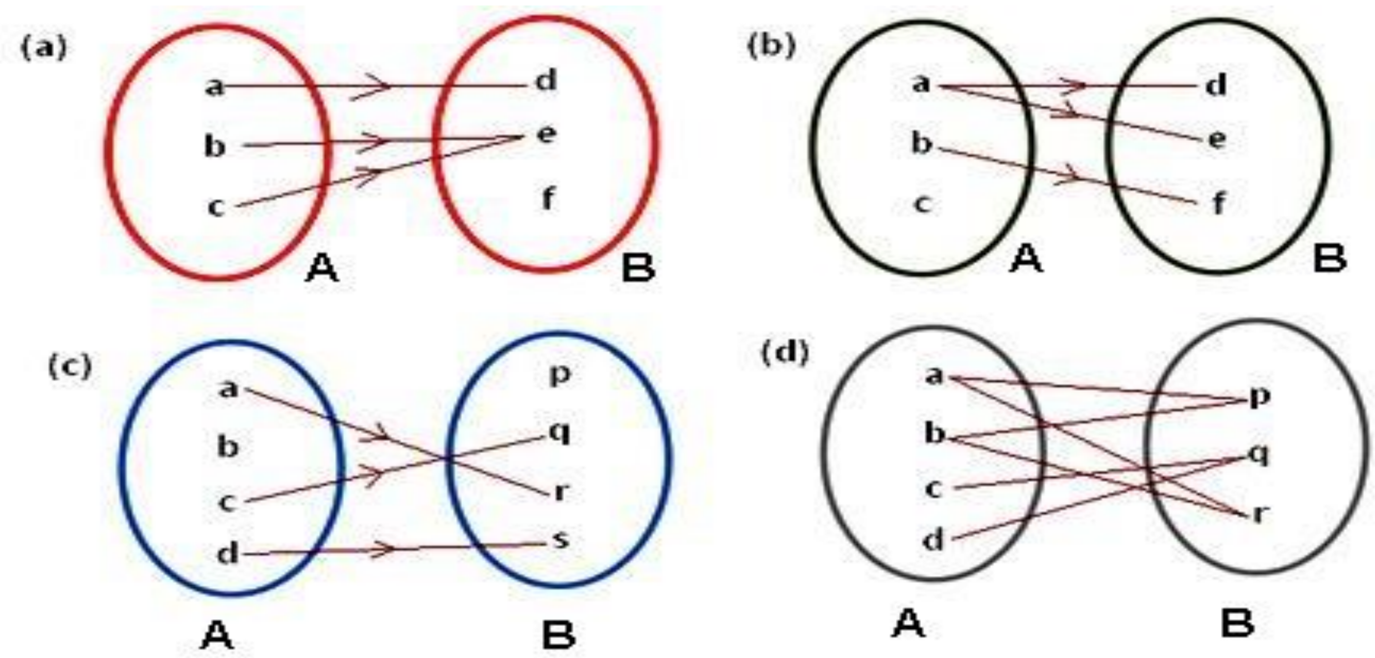
Mapping or Function:

Let A & B are two non-empty sets. A relation f from set A to set B is said to be a mapping or function if – **every** element of set A is associated with **unique** element of set B.

The mapping or function f from set A to set B is denoted by $f : A \to B$ and is read as f maps A into B.

In following four figures, let f be relation from set A to set B. We've to tell which relations are also mappings/functions?



(a) Here f is a mapping / function. We can write f as, $f = \{(a, d), (b, e), (c, e)\}$

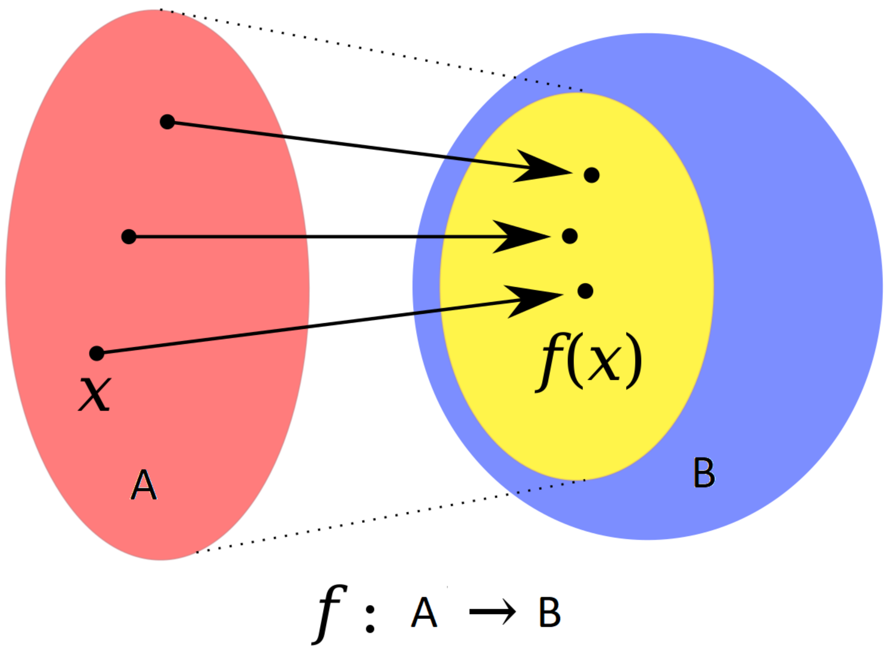
(b) Here f is NOT a mapping for two reasons. The element $a \in A$ is associated with two elements $d, e \in B$. So, element of set A is not associated with *unique* element of B. Secondly, an element $c \in A$ is not associated with any element in B. So, *not every* element of set A is associated with B.

(c) Here f is not a mapping.

(d) Here f is not a mapping.

Image & Pre-image:

In the figure (a) above, $d \in B$ is known as *image* of $a \in A$ under mapping f.



If f is a function from A to B and $x \in A$, then $f(x) \in B$ where $f(x)$ is called the ***image*** of x under f and x is called the ***pre-image*** of $f(x)$ under f.

If f is a mapping from A to B, then every member of set A has unique image within set B, but every member of set B may not have pre-image in A under f.

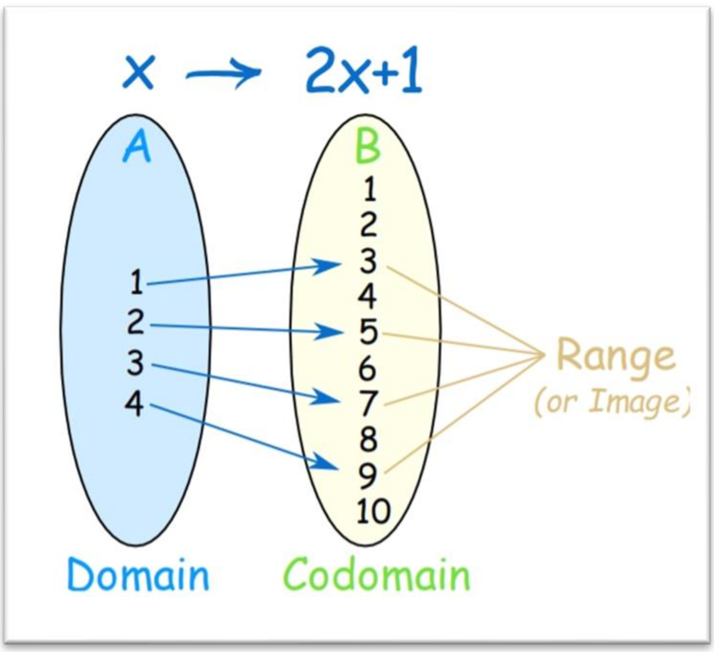
Domain, Co-domain & Range of a function:

Let's start with an example. Let A & B are two non-empty sets given by

$A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6,7, 8, 9, 10\}$.

A mapping $f: A \rightarrow B$ is defined by $f(x) = 2x + 1$.

Question: Find the mapping f in ordered pair set and determine its domain, co-domain & range.



$f = \{(1,3), (2,5), (3,7), (4,9)\}$

Here *domain* of $f = \{1, 2, 3, 4\} = A$

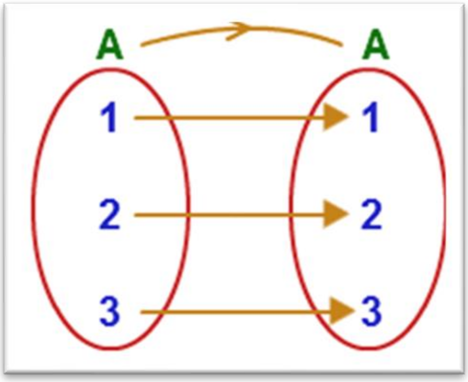
3 is called image of 1 under f.

5 is called image of 2 under f.

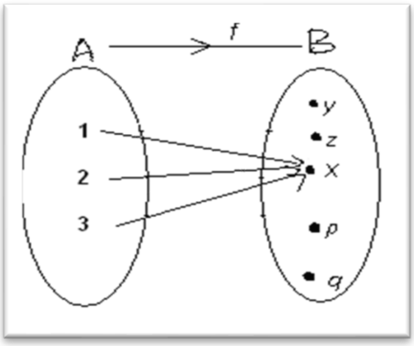
7 is called image of 3 under f.

9 is called image of 4 under f.

The set B is called *co-domain* of mapping f. The set formed by the images of all members of A under mapping f, is called *range* or *image set* of mapping f. Here range of mapping $f = \{3, 5, 7, 9\}$. Note that "Range" is always a subset of co-domain.



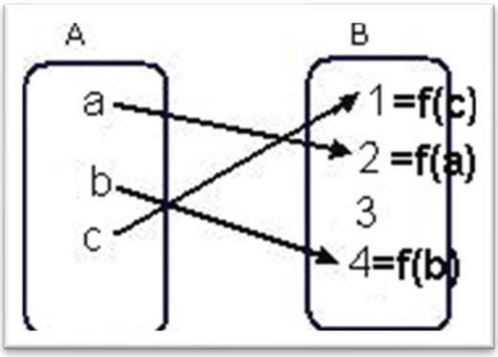
Identity mapping: A mapping $f: A \to A$ is called an Identity mapping if every element of set A is mapped to same element of set B, i.e., $f(x) = x$, $\forall x \in A$



Constant mapping: A mapping $f: A \to B$ is called a constant mapping if every element of set A has same image in set B. The range of this mapping is a singleton set.

Different types of mappings:

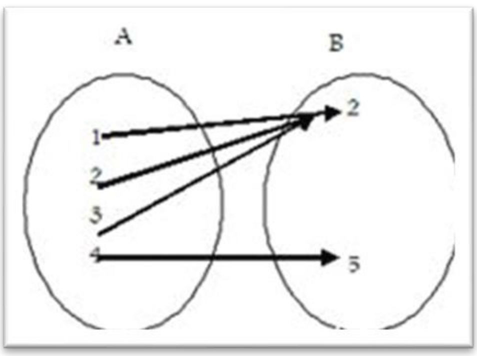
- *Injective mapping / one-to-one mapping / injection*:



A mapping $f: A \to B$ is called an injective mapping if distinct elements of its domain (A) are mapped to distinct elements of its co-domain. Here for all $a, b \in A$

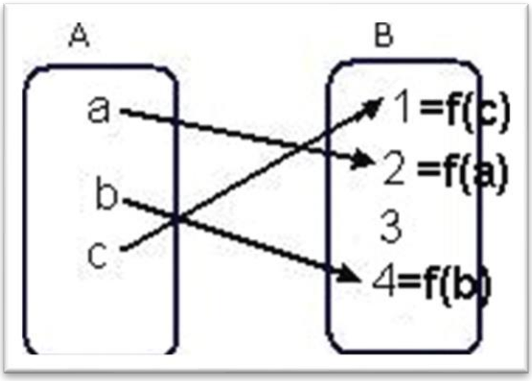
i) $a = b \Rightarrow f(a) = f(b)$
ii) $a \neq b \Rightarrow f(a) \neq f(b)$

- *Many-one mapping*:



A mapping $f: A \to B$ is called a many-one mapping if two or more elements of its domain (A) are mapped to same element of its co-domain (B). Here we see that $1, 2, 3 \in A$ have same image $2 \in B$

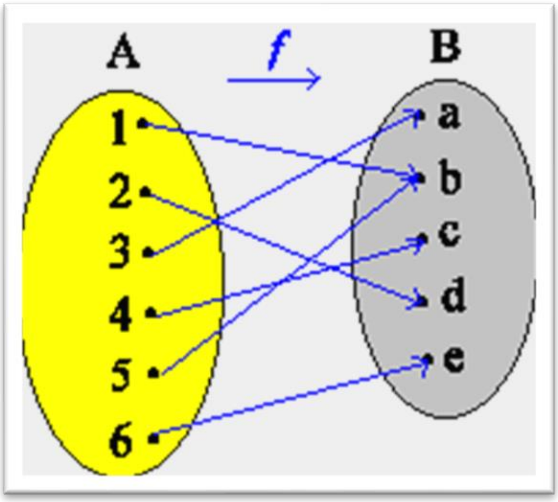
- *Into mapping*:



A mapping $f: A \to B$ is called an into mapping if there exists at least one element in its co-domain (B) which has no pre-image in its domain (A).

Here $3 \in B$ has no pre-image in set A. In this example, range of mapping f is $\{1, 2, 4\}$ which is a subset of co-domain (B).

- *Surjective mapping / onto mapping / surjection*:



A mapping $f: A \to B$ is called a surjective mapping if every element of its co-domain (B) has one/more pre-image in its domain (A).

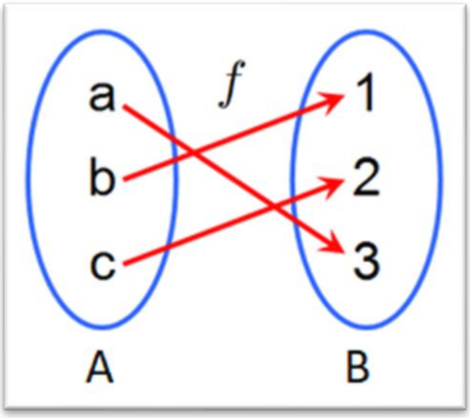
For this mapping, range and co-domain are equal set.

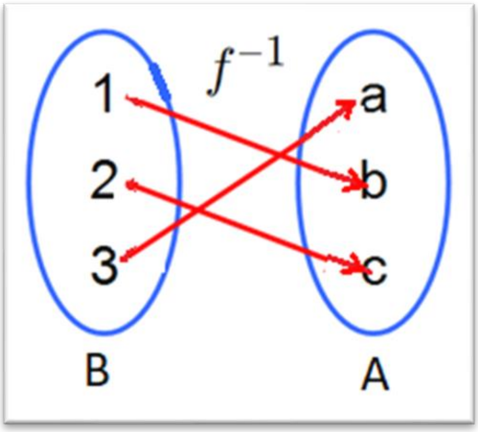
- *Bijective mapping / bijection*:

  A mapping $f: A \to B$ is called a bijective mapping if the mapping is both one-to-one and onto mapping.

E.g. identity mapping is always bijective.

Inverse mapping:





Let $f: A \to B$ is a bijective mapping. Then inverse of f, denoted as f^{-1} maps each element of B to unique element of A. So $f^{-1}: B \to A$ is the inverse mapping of f.

Inverse of mapping exists if and only if the mapping is bijective. i.e., the mapping which is not bijective has no inverse.

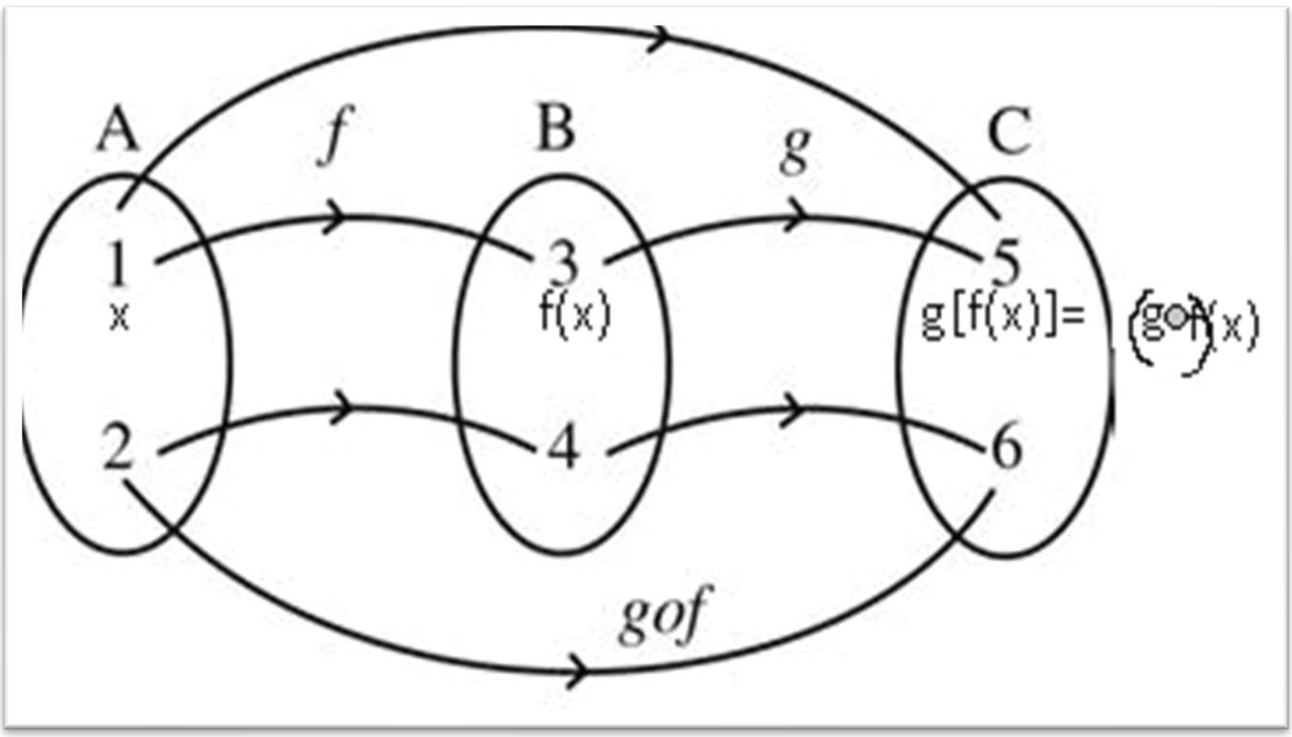
Equality of two mappings:

Two mappings $f: A \to B$ and $g: C \to D$ are said to be equal if

i) domain of f = domain of g i.e., two sets A & C are equal i.e., $A = C$

ii) for all $x \in A$, $f(x) \in B$ & $g(x) \in D$ and $f(x) = g(x)$

Composition of two mappings:



Let A, B, C be three non-empty sets. The composition of two mappings $f: A \to B$ and $g: B \to C$ is denoted by $g \circ f: A \to C$ and is defined by

$$(g \circ f)(x) = g[f(x)] \quad \forall\, x \in A$$

Note: Composition of mappings does not follow commutative law, but follows associative law. That is, if f, g, h are three mappings, then $f \circ g \neq g \circ f$, but $f \circ (g \circ h) = (f \circ g) \circ h$

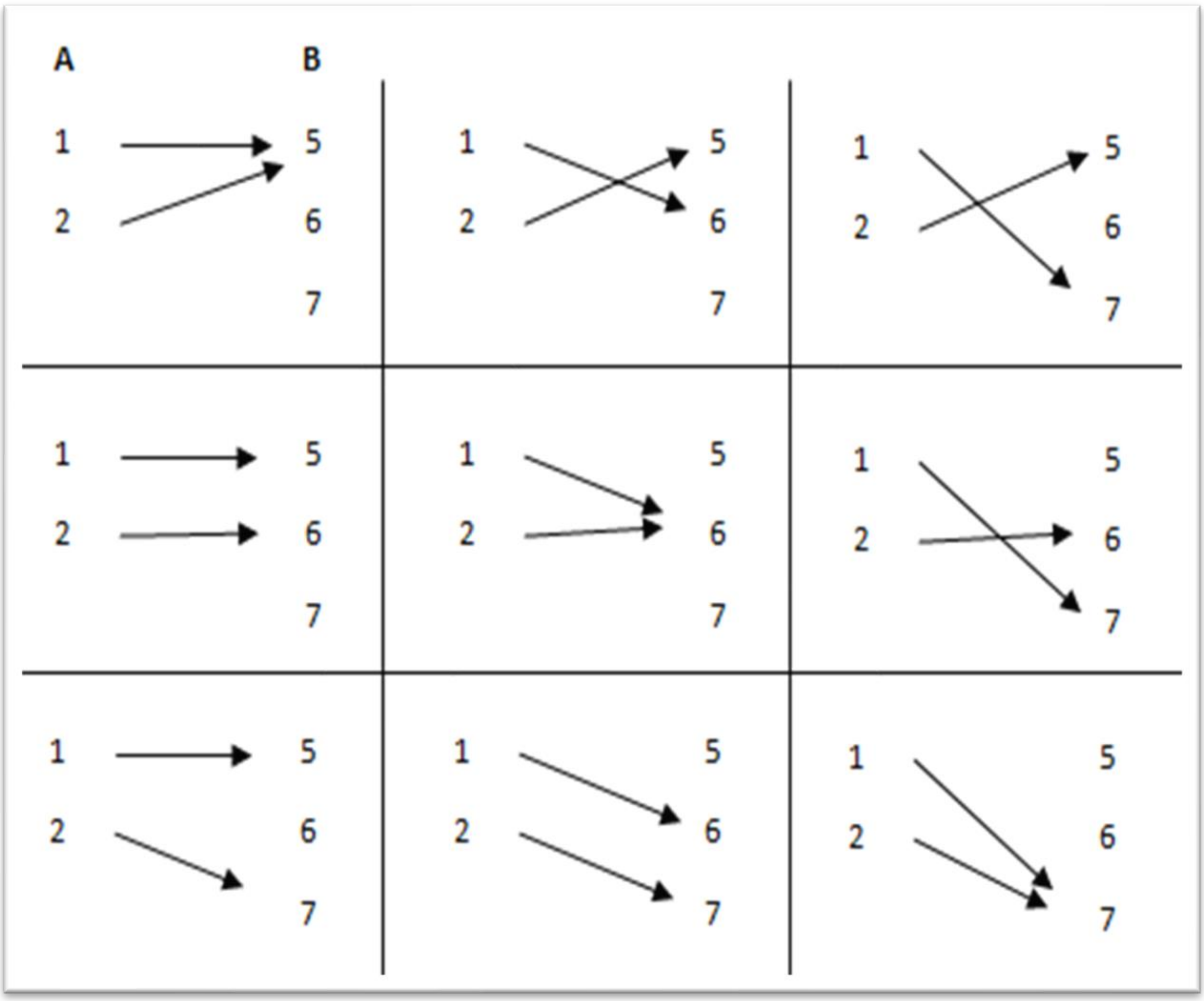
Number of mappings:

Let set A have a elements and set B have b elements. Each element in A has b choices to be mapped to. Each such choice gives you a unique mapping. Since each element has b choices, the total number of mappings from A to B is: $b \times b \times b \cdots \ (a\ times) = b^a$

Now let's see an example.

$A = \{1, 2\}$ & $B = \{5, 6, 7\}$, then $n(A) = 2$ & $n(B) = 3$.

Then distinct mappings $f: A \to B$ are illustrated in the following picture:



1. Given $F(x) = \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} + \frac{(x-a)(x-b)}{(c-a)(c-b)}$; show that $F(0) = 1$

Solution:

$$F(0) = -\frac{bc}{(a-b)(c-a)} - \frac{ca}{(b-c)(a-b)} - \frac{ab}{(c-a)(b-c)}$$

$$= -\left[\frac{bc(b-c)+ca(c-a)+ab(a-b)}{(a-b)(b-c)(c-a)}\right]$$

$$= -\left[\frac{b^2c-bc^2+c^2a-ca^2+a^2b-ab^2}{(a-b)(b-c)(c-a)}\right]$$

$$= -\left[\frac{b^2c-bc^2+c^2a-ca^2+a^2b-ab^2}{(a-b)(bc-ab-c^2+ac)}\right]$$

$$= -\left[\frac{b^2c-bc^2+c^2a-ca^2+a^2b-ab^2}{abc-a^2b-c^2a+ca^2-b^2c+ab^2+bc^2-abc}\right]$$

$$= -\left[\frac{b^2c-bc^2+c^2a-ca^2+a^2b-ab^2}{-(a^2b+c^2a-ca^2+b^2c-ab^2-bc^2)}\right] = 1$$

2. Given $f(x) = \cos(\log x)$; then find the value of $f(x) \cdot f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$

Solution:

$$f(x) \cdot f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$$

$$= \cos(\log x) \cdot \cos(\log y) - \frac{1}{2}\left[\cos\left(\log\frac{x}{y}\right) + \cos(\log xy)\right]$$

$$= \cos(\log x) \cdot \cos(\log y) - \frac{1}{2}[\cos(\log x - \log y) + \cos(\log x + \log y)]$$

$$= \cos(\log x) \cdot \cos(\log y) - \frac{1}{2}[\cos(\log x) \cdot \cos(\log y) + \sin(\log x) \cdot \sin(\log y) + \cos(\log x) \cdot \cos(\log y) - \sin(\log x) \cdot \sin(\log y)]$$

$$= 0$$

3. If $f(x) = ax^2 + bx + c$, then find the value of a, b such that $f(x+1) = f(x) + x + 1$ be an identity.

Solution:

$f(x+1) = f(x) + x + 1$

$\Rightarrow a(x+1)^2 + b(x+1) + c = ax^2 + bx + c + x + 1$

$\Rightarrow ax^2 + 2ax + a + bx + b + c = ax^2 + bx + x + c + 1$

$\Rightarrow ax^2 + (2a+b)x + (a+b+c) = ax^2 + (b+1)x + (c+1)$

As it is an identity, we can compare the coefficients of x^2, x and constant terms of both sides.

$\therefore 2a + b = b + 1 \Rightarrow a = \frac{1}{2}$

$\therefore a + b + c = c + 1 \Rightarrow a + b = 1 \Rightarrow b = 1 - a = \frac{1}{2}$

4. If $y = f(x) = \frac{3x-5}{2x-m}$, then find the value of m such that $f(y) = x$

Solution:

$f(y) = x$

$\Rightarrow \frac{3y-5}{2y-m} = x$

$\Rightarrow \frac{3 \cdot \frac{3x-5}{2x-m} - 5}{2 \cdot \frac{3x-5}{2x-m} - m} = x$ (putting the value of y, given)

$\Rightarrow \frac{9x-15-10x+5m}{6x-10-2mx+m^2} = x$

$\Rightarrow -x - 15 + 5m = 6x^2 - 10x - 2mx^2 + m^2x$

$\Rightarrow -x - 15 + 5m = (6-2m)x^2 + (m^2-10)x$

To find the value of m, we compare the coefficient of x^2, x and constant terms of both sides of above equation.

$\therefore 6 - 2m = 0 \Rightarrow m = 3$

$\therefore m^2 - 10 = -1 \Rightarrow m^2 = 9 \Rightarrow m = \pm 3$

$\therefore -15 + 5m = 0 \Rightarrow m = 3$

So, acceptable value of $m = 3$

5. Assume that a function $f: ? \to \mathbb{R}$, defined by the following rules. Find domain of definitions in each case.

i) $\sqrt{x^2 - 7x + 10}$

ii) $\sqrt{4x - 4x^2 - 1}$

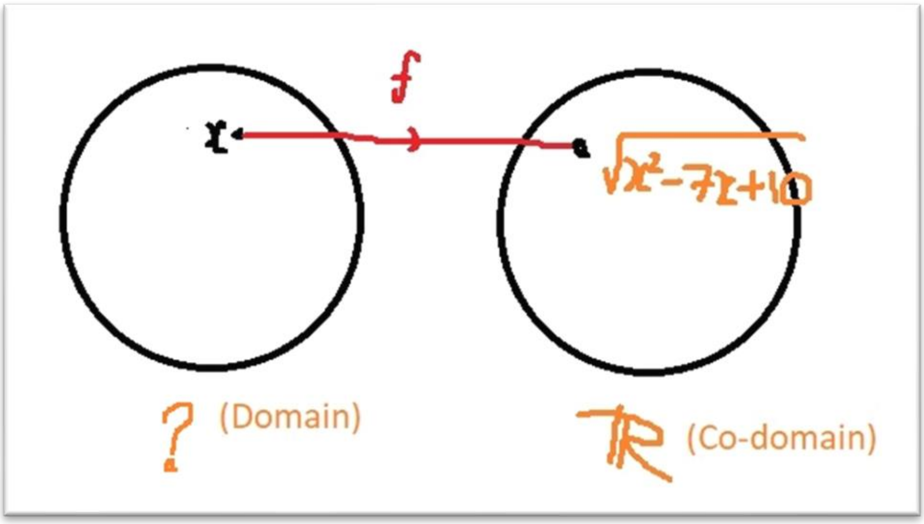
iii) $\sqrt{x^2 - 4x + 3}$

iv) $\frac{x^2}{1+x^2}$

v) $\frac{1}{\sin x - \cos x}$

Solutions:

i) $f(x) = \sqrt{x^2 - 7x + 10}$



We see that, $f(x) \in \mathbb{R}$ (co-domain), if $x^2 - 7x + 10 \geq 0$, where $x \in$ Domain

So, $x^2 - 7x + 10 \geq 0$

$\Rightarrow \ (x - 5)(x - 2) \geq 0$

This is possible if –

- $(x - 5) \geq 0$ & $(x - 2) \geq 0$
- $(x - 5) \leq 0$ & $(x - 2) \leq 0$

From first case, we get $x \geq 5$ and from second case, $x \leq 2$

So, Domain = $\{x \mid x \in \mathbb{R}\ and\ x \leq 2\ or\ x \geq 5\}$

ii) $f(x) = \sqrt{4x - 4x^2 - 1}$

We see that, $f(x) \in \mathbb{R}$ (co-domain), if $4x - 4x^2 - 1 \geq 0$, where $x \in$ Domain

So, $4x - 4x^2 - 1 \geq 0$

$\Rightarrow -(4x^2 - 4x + 1) \geq 0$

$\Rightarrow (4x^2 - 4x + 1) \leq 0$

$\Rightarrow (2x - 1)^2 \leq 0$

$\Rightarrow (2x - 1) = 0$ (As, square quantity is always positive)

$\Rightarrow x = \frac{1}{2}$

So, Domain = $\left\{\frac{1}{2}\right\}$

iii) $f(x) = \sqrt{x^2 - 4x + 3}$

We see that, $f(x) \in \mathbb{R}$ (co-domain), if $x^2 - 4x + 3 \geq 0$, where $x \in$ Domain

So, $x^2 - 4x + 3 \geq 0$

$\Rightarrow (x - 1)(x - 3) \geq 0$

This is possible if –

- $(x - 1) \geq 0$ & $(x - 3) \geq 0$
- $(x - 1) \leq 0$ & $(x - 3) \leq 0$

From first case we get $x \geq 3$ and from second case $x \leq 1$

So, Domain = $\{x \mid x \in \mathbb{R}\ and\ x \leq 1\ or\ x \geq 3\} = (-\infty, 1] \cup [3, \infty)$

iv) $f(x) = \frac{x^2}{1+x^2}$

We see that, $f(x) \in \mathbb{R}$ (co-domain), if $1 + x^2 \neq 0$, where $x \in$ Domain

But, for all $x \in \mathbb{R}$, $x^2 \neq -1$

So, Domain = $\{x \mid x \in \mathbb{R}\}$

v) $f(x) = \frac{1}{\sin x - \cos x}$

We see that, $f(x) \in \mathbb{R}$ (co-domain), if $\sin x - \cos x \neq 0$, where $x \in$ Domain

So, $\sin x - \cos x \neq 0$

$\Rightarrow \sin x \neq \cos x$

$\Rightarrow \tan x \neq 1$

We know that, all trigonometric functions are periodic functions.

Now, $\tan \frac{\pi}{4} = 1$

$\Rightarrow \tan\left(2n \cdot \frac{\pi}{2} + \frac{\pi}{4}\right) = 1$ [$\because \tan(n\pi + \theta) = \tan\theta$ when n is even integer]

So, Domain = $\left\{x \mid x \in \mathbb{R}\ and\ x \neq n\pi + \frac{\pi}{4}\ where\ n\ is\ any\ integer\right\}$

6. If $f(x) = \tan^{-1} x$, find the relation by which $f(x)$, $f(y)$ and $f(x+y)$ are connected.

Solution:

$f(x) = \tan^{-1} x \Rightarrow \tan\{f(x)\} = x$

$f(y) = \tan^{-1} y \Rightarrow \tan\{f(y)\} = y$

$f(x+y) = \tan^{-1}(x+y)$

$\Rightarrow \tan\{f(x+y\} = x + y = \tan\{f(x)\} + \tan\{f(y)\}$ (From first two equations)

7. If $f(x) = \frac{x}{x+1}$, $g(x) = x^{10}$ and $h(x) = x + 3$; then find $f \circ g \circ h$

Solution:

We can write $f \circ g \circ h = f \circ (g \circ h)$

Now, $g \circ h = (g \circ h)(x) = g[h(x)] = g(x+3) = (x+3)^{10}$

Now, $f \circ (g \circ h) = f[(x+3)^{10}] = \frac{(x+3)^{10}}{(x+3)^{10}+1}$

8. If f is an even function and g is odd function, then the function $f \circ g$ is:

(a) even function, (b) odd function, (c) neither even nor odd

Solution: As $f(x)$ is an even function, then $f(-x) = f(x)$

& as $g(x)$ is an odd function, then $g(-x) = -g(x)$

We know that,$\{f \circ g\}(x) = f(g(x))$

So, $\{f \circ g\}(-x) = f(g(-x)) = f(-g(x))$ [As g is an odd function]

$= f(-y)$ [Let, $g(x) = y$]

$= f(y)$ [As, f is an even function]

$= f\{g(x)\}$ [As, $y = g(x)$]

$= (f \circ g)(x)$

So, $f \circ g$ is an even function.

9. For a real function $f(x)$ is defined by $f(x) = \sqrt{x-2}$, find $(f \circ f \circ f)(38)$

Solution: $(f \circ f \circ f)(38) = (f \circ f)\{f(38)\}$ [As composition of mapping follows associative property]

$= (f \circ f)\left\{\sqrt{38-2}\right\}$

$= (f \circ f)(6) = f\{f(6)\} = f\left\{\sqrt{6-2}\right\} = f(2) = \sqrt{2-2} = 0$